Causal Inference in Statistics
Chapter 3 : Interventions

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Overview

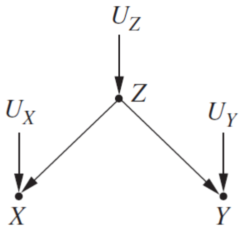
1. What is Interventions
2. The way to get causal effect
   - Adjustment Formula
   - Backdoor Criterion
   - Front-Door Criterion
   - Covariate-Specific Effects
   - IP-Weighting
   - Mediation
   - Linear System

Section 1

What is Interventions

Association vs Causation

- Under **randomized experiments**, it is very easy to verify causation.
- However, it is difficult to conduct randomized experiments in many cases, so researchers perform **observational studies** instead.
- In observational studies, **association** and **causation** are different.



Conditioning vs Intervening

- (X=x) means **Conditioning** : focus on partial samples
- $P(Y = y|X = x) = \frac{\text{\# of (X=x, Y=y)}}{\text{\# of (X=x)}} \rightarrow$ Association

- do(X=x) means **Intervention**: focus on all samples
- $P(Y = y|do(X = x)) = \frac{\text{\# of (Y=y) if samples were (X=x)}}{\text{\# of samples}} \rightarrow$ Causation

Conditioning vs Intervening

- In causal inference, our main interest is $P(Y = y|do(X = x))$ (**causal effect**).
- But we can only get $P(Y = y|X = x)$, and it is different with $P(Y = y|do(X = x))$ in observational study.
- In this chapter, our purpose is to get $P(Y = y|do(X = x))$ from $P(Y = y|X = x)$

Section 2

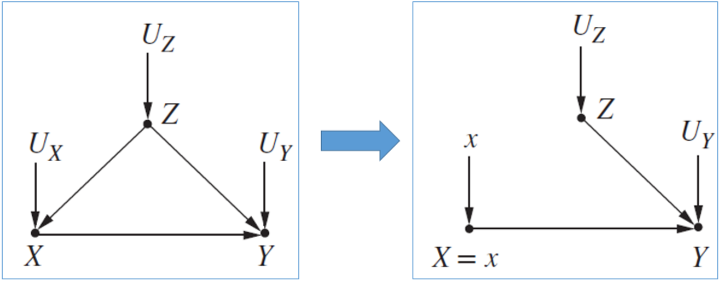
The way to get causal effect

The Causal Effect Rule 1

Given a graph G in which a set of variables PA are designated as the parents of X, the causal effect of X on Y is given by

$$P(Y = y|do(X = x)) = \sum_z P(Y = y|X = x, PA = z)P(PA = z). \quad (1)$$

- When we intervene X to x, all edges directed into X are removed, and all samples's X become x.
- Let original graph as G, new graph as G_m



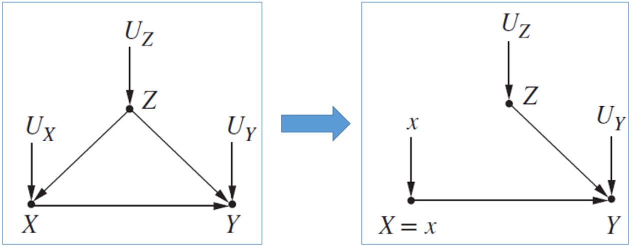
The Adjustment Formula : Derivation

$$P(Y = y|do(X = x)) = P_m(Y = y|X = x)$$

$$= \sum_z P_m(Y = y|X = x, Z = z)P_m(Z = z|X = x)$$

$$= \sum_z P_m(Y = y|X = x, Z = z)P_m(Z = z)$$

$$= \sum_z P(Y = y|X = x, Z = z)P(Z = z)$$



The Backdoor Criterion

- Sometimes it is too hard to measure all parents, so conditioning is also impossible.
- We want to find set of variable to **replace** set of parents.

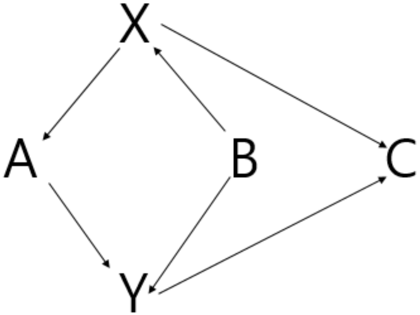
The Backdoor Criterion

Given an ordered pair of variables (X, Y) in a directed acyclic graph G, a set of variables Z satisfies the backdoor criterion relative to (X, Y) if

1. No node in Z is a descendant of X
2. Z **blocks every path between** X **and** Y **that contains an arrow into** X

d-separation vs The Backdoor Criterion

For example, in the graph below...



- X and Y are **d-separated** conditional on $Z_1 = \{A, B\}$
- $Z_2 = \{B\}$ satisfies **the backdoor criterion** relative to (X,Y)

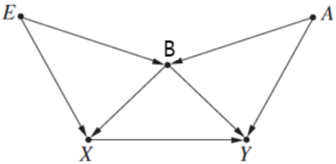
The Causal Effect Rule 2

Given a graph G in which a set of variables Z satisfies the backdoor criterion for X and Y, the causal effect of X on Y is given by

$$P(Y = y|do(X = x)) = \sum_z P(Y = y|X = x, Z = z)P(Z = z). \quad (2)$$

The Backdoor Criterion

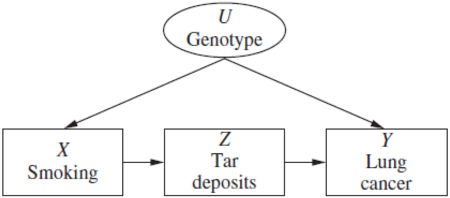
Example



- There are four paths that contains an arrow into X, and we have to block all of them.
- For block $X \leftarrow B \rightarrow Y$, we have to adjust for B
- Then the path $X \leftarrow E \rightarrow B \leftarrow A \rightarrow Y$ becomes open, so we have to adjust for E(or A) also.
- Then $Z = \{B, E\}$(or $Z = \{A, B\}$, $Z = \{A, B, E\}$) satisfies backdoor criterion.

The Front-Door Criterion

- There are situations in which we can apply the Back-Door criterion *provided* a set of variables are measurable in form of actual data.
- Otherwise, if we cannot measure those variables, we may consider another method called the Front-Door criterion.
- For example, we can't measure U(Geneotype), but can measure Z(Tar deposits) instead.

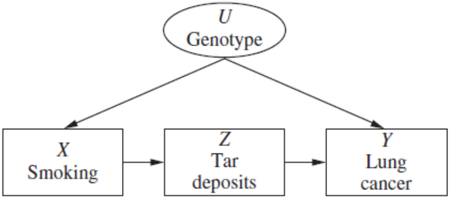


- First, we look at the pair X and Z. Since there is no backdoor path from X to Z, we have

$$P(Z = z|do(X = x)) = P(Z = z|X = x)$$

- Next, if we see that there is a backdoor path from Z to Y, and it can be blocked by X. Thus, by the causal effect rule,

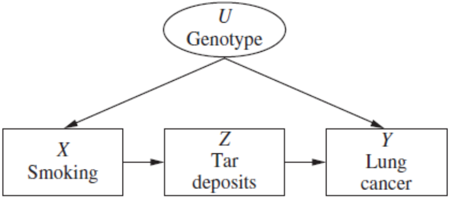
$$P(Y = y|do(Z = z)) = \sum_{x'} P(Y = y|Z = z, X = x')P(X = x')$$



- Combining two results, we get what is called **"the Front-Door Formula"**:

$$P(Y = y|do(X = x)) = \sum_z P(Y = y|do(Z = z))P(Z = z|do(X = x))$$

$$\sum_z \sum_{x'} P(Y = y|Z = z, X = x')P(X = x')P(Z = z|X = x)$$



The Front-Door Criterion

In general, the textbook states the following definition and Theorem regarding the Front-Door Criterion.

Definition 3.4.1(Front-Door) *A set of variables Z is said to satisfy the front-door criterion relative to an ordered pair of variables (X, Y) if*

1. Z intercepts all directed paths from X to Y.
2. There is no unblocked path from X to Z except $X \to Z$.
3. All backdoor paths from Z to Y are blocked by X.

Theorem 3.4.1(Front-Door) *If Z satisfies the front-door criterion relative to (X, Y), then the causal effect of X on Y is identifiable and is given by the formula*

$$P(y|do(x)) = \sum_z P(z|x) \sum_{x'} P(y|x', z)P(x')$$

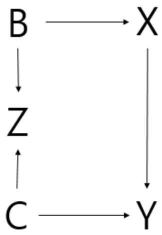
- Sometimes, we want to find **"z-specific effect" of X**, namely, $P(Y = y|do(X = x), Z = z)$. This is the distribution of Y conditioned on Z *after* the intervention on X.

Rule 3 *The specific effect $P(Y = y|do(X = x), Z = z)$ is identified whenever we can measure a set S of variables such that $S \cup Z$ satisfies the backdoor criterion. Moreover, the z-specific effect is given by the following adjustment formula*

$$P(Y = y|do(X = x), Z = z)$$
$$= \sum_s P(Y = y|X = x, S = s, Z = z)P(S = s|Z = z)$$

Covariate-Specific Effects

Example



- If we want to compute the causal effect of X on Y for a specific value z of Z, we must adjust for B(or C) additionally.
- $P(y|do(x), z) = \sum_b P(y|x, b, z)P(b|z)$

Inverse Probability Weighting

- Using adjustment formula involves computing probabilities conditioned on Z.
- We may hit a case in which the computation is burdensome.
- For example, if Z is a categorical variable and the number of the feature of Z is greater than the size of the data at hand, computation work might be heavy and the quality of estimation may be poor.

Inverse Probability Weighting

- However, observing the follow equality provides us a detour:

$P(y|do(x)) = \sum_z P(Y = y|X = x, Z = z)P(Z = z)$

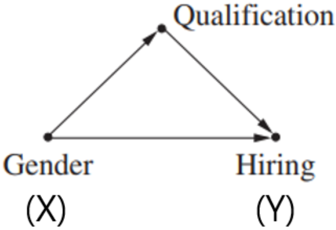
$= \sum_z \frac{P(Y=y|X=x,Z=z)P(X=x|Z=z)P(Z=z)}{P(X=x|Z=z)}$

$= \sum_z P(Y = y, X = x, Z = z)/P(X = x|Z = z)$

- So long as we can find a good estimate of $P(X = x|Z = z)$, we can simply estimate $P(y|do(x))$ by substituting $P(X = \hat{x}|Z = z)$ for $P(X = x|Z = z)$, which is in the denominator part of the formula.

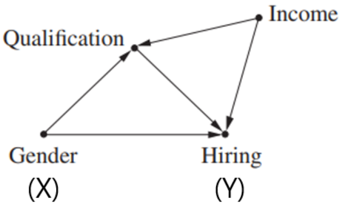
Mediation

- There are cases in which we want to separate the direct effect of a variable on another variable from the indirect one.
- This is usually achieved by conditioning on *mediating variables* between the two variables of our interest, which derive the indirect effect of a variable on another. See the figure below. The variable Qualification is a mediating variable of X and Y in this system.



Mediation

- However, there could be other variables which may make this conditioning work undesirable. Consider the case below, where we also need to consider Income variable.
- Since Qualification is now a collider, conditioning on this variable culminates in making another indirect path from X to Y, which nullify our effort to distinguish between the direct and indirect effect of X onto Y.

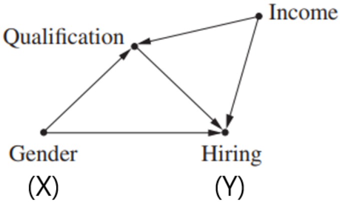


Mediation

- Resolving this problem requires us to make intervention on the both variables Qualification(mediating variable) and Income(confounding variable).
- Thus we are looking for what is called the *controlled direct effect*(CDE) on Y of changing the value of X from x to x' when Q is a mediator between X and Y.

$$CDE(X, Y; Q) =$$

$$P(Y = y|do(X = x), do(Q = q)) - P(Y = y|do(X = x'), do(Q = q))$$



Mediation

- To find $CDE(X, Y; Q)$ in this case, first note that we have no arrows onto Q after the intervention on Q. Hence, there is no backdoor path from X to Y, which gives:

  $CDE(X, Y; Q) =$

  $$P(Y = y|X = x, do(Q = q)) - P(Y = y|X = x', do(Q = q))$$

- After then, we simply apply adjustment formula only taking the variable I(income) into account, since X is already blocked by conditioning.

  $CDE(X, Y; Q) = \sum_i [P(Y = y|X = x, Q = q, I = i)$
  $-P(Y = y|X = x', Q = q, I = i)]P(I = i)$

- Until now, methods work regardless of the type of equations that make up the model in question.**(i.e. nonparametric)**

- However, explaining causal methods from a nonparametric standpoint has **limited our ability** to present the full power of these methods as they generate in linear systems

- In this section, we examine what causal assumptions and implications look like in systems of **linear equations**

Causal Inference in Linear Systems

- We will assume
  1. Relationships between variables are linear
  2. All error terms have normal distribution

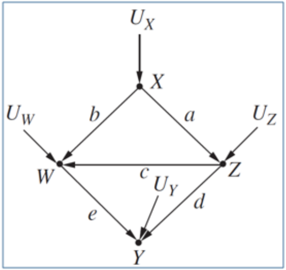
- For example,

$$X = U_X$$

$$Z = aX + U_Z$$

$$W = bX + cZ + U_W$$

$$Y = dZ + eW + U_Y$$



Causal Inference in Linear Systems

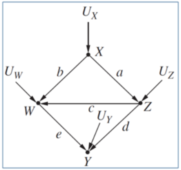
Direct effect

- **Direct effect** of Z on Y : the change in Y as Z increases by one unit whereas all other parents of Y are held constant.
- In linear system, every path coefficient stands for the direct effect.

Total effect

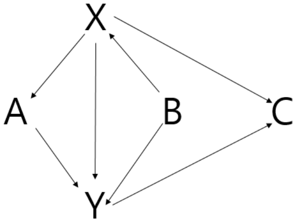
- **Total effect** of Z on Y : the change in Y as Z increases by one unit(other parents of Y can change).
- First, find every directed path from Z to Y; then, for each path, multiply all coefficients on the path, and add up all the products.

$$X = U_X$$
$$Z = aX + U_Z$$
$$W = bX + cZ + U_W$$
$$Y = dZ + eW + U_Y$$



Identify Direct Effect of X on Y

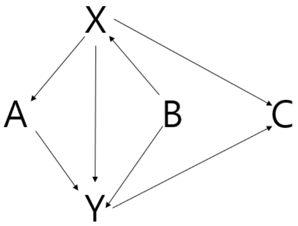
1. Remove the edge from X to Y (if exists), and call the resulting graph G_α.
2. Let Z be a set of variables that d-separates X and Y on G_α.
3. Regress Y on X and Z, then coefficient of X represents the direct effect



- 3 path between X and Y in G_α
- $Z = \{A, B\}$
- $Y = r_0 + r_1 X + r_2 A + r_3 B$
- Then, we estimate r_1 as direct effect

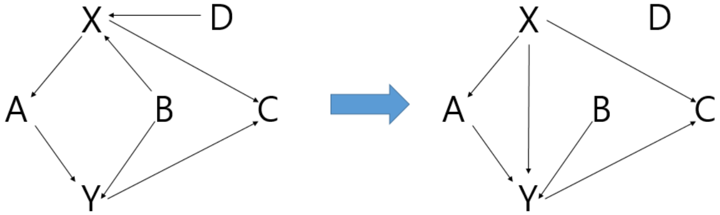
Identify Total Effect of X on Y

1. Find a set of covariates Z that satisfies the backdoor criterion from X to Y in the model.
2. Regress Y on X and Z, then coefficient of X represents the total effect



- $Z = \{B\}$ satisfies backdoor criterion
- $Y = r_0 + r_1 X + r_2 B$
- Then, we estimate r_1 as total effect

- When we intervene X to x, all edges directed into X are removed, and all samples's X become x.
- Let original graph as G, new graph as G_m
- Let Z satisfies the backdoor criterion from X to Y in the model.



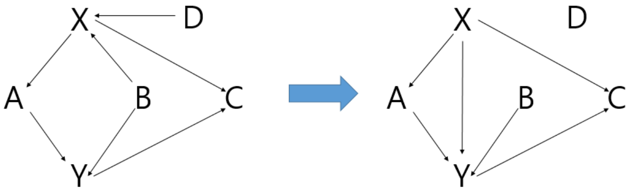
The Backdoor Criterion : Derivation

$$P(Y=y|do(X=x)) = P_m(Y=y|X=x)$$

$$= \sum_z P_m(Y=y|X=x, Z=z)P_m(Z=z|X=x)$$

$$= \sum_z P_m(Y=y|X=x, Z=z)P_m(Z=z)$$

$$= \sum_z P(Y=y|X=x, Z=z)P(Z=z)$$



Covariate-Specific Effects : Derivation

$P(Y = y|do(X = x), Z = z) = P_m(Y = y|X = x, Z = z)$

$= \sum_s P_m(Y = y|X = x, Z = z, S = s)P_m(S = s|X = x, Z = z)$

$= \sum_s P_m(Y = y|X = x, Z = z, S = s)P_m(S = s|Z = z)$
(X and Z are independent after the intervention)

$= \sum_s P(Y = y|X = x, Z = z, S = s)P(S = s|Z = z)$
(No backdoor path after conditioning on S.)